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STOCHASTIC INVERSE PROBLEM IN THE RADIATION OF NOISE*

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Abstract. A general procedure for regularizing the stochastic inverse source problem is presented. The present approach, based on the minimization of a structural functional, allows the incorporation of a priori knowledge about the structure of the undetermined source. As special cases, it yields the methods of Lagrangian multipliers, Tihonov's regularization and the pseudoinverses. Possible generalization and connection with Franklin's well-posed extensions are discussed briefly.

1. Introduction. This paper is concerned with an inverse problem in the radiation from a randomly fluctuating source. The problem was suggested by a study on the determination of the unknown source distribution in a jet from the far-field acoustic radiation data. According to Lighthill's model [1], the jet noise is generated by the fluctuating part of the fluid velocities in the jet. In this case, the pressure field is governed by an inhomogeneous wave equation, where the source term is given by the second order divergence of the Reynolds stress tensor. A great majority of works [2] in jet noise is concerned with finding the pressure correlation function by specifying the source covariance function, e.g. a quadrupole of certain form. In reality, the source is neither known a priori, nor directly measurable. It is the acoustic pressure fluctuation at a given point that can be measured by a microphone. Question naturally arises as to the possibility of using a set of acoustic radiation data to infer the characteristics of the source distribution. This gives rise to a stochastic inverse problem.

The common mathematical difficulty in dealing with inverse problems in physical sciences lies in the fact that they are generally ill-posed in the Hadamard sense [3]. Recall that a mathematical problem is well-posed if the problem has a solution, the solution is unique and it depends continuously on data. The inverse source problem is clearly ill-posed, because a set of radiation data, say, on a surface enclosing the source region is insufficient to determine the source distribution. However the partial and inaccurate data should be of value in the reconstruction of source.

Conceivably there are many possible ways to utilize the radiation data to yield some information about the source. A procedure that, by introducing a priori assumptions or otherwise, renders a unique, stable solution to an ill-posed problem is called a regularization or a well-posed extension. Most of regularization procedures clusters around the well-known method of least squares, notably Tihonov's regularization [4] the well-posed extension by Franklin [5] and others [6], [7]. Though the approaches are different, these methods for regularizing deterministic problems are not totally unrelated. The present problem is stochastic in nature and, hence, there is need for a generalization of the current procedures. However a full generalization will not be done here. For tractability, our analysis will be confined to the estimation of the source covariance, known as the correlation theory in engineering and science. Then, within this theory, we are able to adapt a generalized, deterministic method to the stochastic problems. The regularization procedures of Tihonov and others are essentially stabilization schemes without taking the underlying physical processes into consideration. To reflect the common sense that more information being given about

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the physical problem should reduce the degree of uncertainty, we shall propose a variational regularization procedure which allows the incorporation of a priori knowledge about the structure of the undetermined source. Our procedure resembles the common practice of introducing an extremal principle in continuum physics to characterize the physically acceptable solution among admissable ones. To this end, a structural functional and a residual functional depending on a parametric function λ will be introduced. The sum of these two will be termed the Lagrangian functional. Then our regularization procedure consists of minimizing the Lagrangian functional, with or without constraints. It is believed that the proposed method is more flexible and better adapted to the physical problem under consideration.

To be specific, this paper will be concerned with a stochastic inverse radiation problem in a uniform medium. The formulation and a preliminary analysis of the inverse problem are given in § 2. As an example, a simple model consisting of an array of point sources is presented and analyzed in § 3. Here the entropy functional is chosen to be the structural functional in determining the source distribution. In the following section, a general theory for the stochastic inverse problem is introduced. For computational feasibility, our emphasis will be placed on quadratic structural functionals. It will be shown that, by specializing the structural and residual functionals, the general procedure yields the methods of Lagrangian multiplier [8], Tihonov's regularization and the method related to the generalized or pseudoinverses [9]. This is done in § 5. In the Appendix, a possible generalization to other radiation problems and a modified version of Franklin's stochastic extension are briefly discussed.

Inverse problems arise from all areas of engineering and physical sciences. The well-known examples are the determination of historic climate by solving the heat equation backward, finding the mass distribution in the earth by measuring the gravitational potential, the inverse scattering problems in the classical and quantum mechanics, etc. References to these problems can be found in the books [10]–[12]. An informal and interesting introduction to the subject of inverse problems was given by Keller [13]. The nature of nonuniqueness of solutions poses a much more serious question [14], which will be our major concern. Therefore a regularized problem, by convention in this paper, will often refer to a problem with a unique solution. The continuous dependence question can be settled separately by an additional smoothing procedure, if necessary.

2. Formulation and preliminaries. Consider the acoustic radiation problem in a uniform medium due to a randomly fluctuating source. Let the source distribution $\tilde{q}(\bar{x}, t, \omega)$ be a random function defined in a region D in $\mathbb{R}^3$, time $t \geqq 0$ and $\omega \in \Omega$ being the underlying sample space. Then the fluctuating pressure field $\tilde{p}(\bar{x}, t, \omega)$ satisfies the inhomogeneous wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\tilde{p} = \tilde{q}, \qquad \bar{x} \in \mathbb{R}^3, \quad t > 0, \tag{2.1}$$

where the speed of propogation c is constant and $\tilde{q} = 0$ if $\bar{x} \notin D$. In a theory of jet noise, $\tilde{q} = (\partial^2/\partial x_j \cdot \partial x_k)T_{jk}$, where T_{jk} is known as the Lighthill's turbulent stress tensor and the summation convention is in effect. Given $\tilde{q}$, the pressure field $\tilde{p}$ in (2.1) is obtained in terms of the retarded potential [15]

$$\tilde{p}(\bar{x}, t, \omega) = \frac{1}{4\pi}\int_D \frac{\tilde{q}(\bar{\xi}, t - (r/c), \omega)}{r}\, d\bar{\xi}, \tag{2.2}$$

where $r = |\bar{x} - \bar{\xi}|$.

For a smooth random function $f(\bar{x}, t, \omega)$, its ensemble average (or mathematical expectation) over the sample space Ω is denoted by $\langle f\rangle$. Assuming the source $\tilde{q}$ being time-stationary, the mean fields $\langle\tilde{q}\rangle$ and, hence, $\langle\tilde{p}\rangle$ are independent of time. They may be taken to be zeros without affecting the acoustic signal fluctuations. Then the covariance of $\tilde{p}$ and $\tilde{q}$ become simply

$$\tilde{P}(\bar{x}, \bar{y}, \tau) = \langle\tilde{p}(\bar{x}, t)\tilde{p}(\bar{y}, t+\tau)\rangle, \tag{2.3}$$

$$\tilde{Q}(\bar{x}, \bar{y}, \tau) = \langle\tilde{q}(\bar{x}, t)\tilde{q}(\bar{y}, t+\tau)\rangle. \tag{2.4}$$

In view of (2.2), the following integral equation can be derived

$$\tilde{P}(\bar{x}, \bar{y}, \tau) = \left(\frac{1}{4\pi}\right)^2 \int_{D^2} \frac{\tilde{Q}[\bar{\xi}, \bar{\eta}, \tau - c^{-1}(r-s)]}{rs}\, d\bar{\xi}\, d\bar{\eta}, \qquad \bar{x}, \bar{y} \in \mathbf{R}^3, \quad -\infty < \tau < \infty, \tag{2.5}$$

where $D^2 = D \times D$, $r = |\bar{x} - \bar{\xi}|$, $s = |\bar{y} - \bar{\eta}|$.

Now let S_ρ be a spherical surface enclosing the region D with radius ρ. Suppose that the radiation data $\tilde{P}$ is given for all $\bar{x}, \bar{y} \in S_\rho$ and $|\tau| < \infty$ as $\rho \to \infty$, (in reality, only a finite, discrete set of data is obtainable). One is required to determine the covariance function $\tilde{Q}$ in D based on an incomplete and inaccurate set of data on $S = S_\rho$ as $\rho \to \infty$. As it stands, the problem is ill-posed. Aside from the question of continuous dependence, there may be too many solutions.

To simplify the relation (2.5) in the far field, we assume that the spectral density functions P, Q exist. They are defined as the Fourier transforms of $\tilde{P}$, $\tilde{Q}$ in τ with the parameter (kc):

$$P(\bar{x}, \bar{y}, k) = \int \tilde{P}(\bar{x}, \bar{y}, \tau)\, e^{-ikc\tau}\, d\tau, \tag{2.6}$$

$$Q(\bar{x}, \bar{y}, k) = \int \tilde{Q}(\bar{x}, \bar{y}, \tau)\, e^{-ikc\tau}\, d\tau. \tag{2.7}$$

Then we have $P^*(\bar{x}, \bar{y}, k) = P(\bar{y}, \bar{x}, k) = P(\bar{x}, \bar{y}, -k)$ and $P(\bar{x}, \bar{x}, k) = P(\bar{x}, \bar{x}, -k) \geqq 0$, where $*$ means the complex conjugate. Similar relations hold for Q.

Noting (2.6) and (2.7), with a Fourier transform of (2.5), we obtain

$$P(\bar{x}, \bar{y}, k) = \int_{D^2} G(r, k) G^*(s, k) Q(\bar{\xi}, \bar{\eta}, k)\, d\bar{\xi}\, d\bar{\eta}, \tag{2.8}$$

where

$$G(r, k) = \frac{e^{ikr}}{4\pi r} \tag{2.9}$$

is the free-space Green's function.

If $|\bar{x}| = |\bar{y}| = \rho$, as $\rho \to \infty$, we get

$$r = |\bar{x} - \bar{\xi}| \sim \rho - \bar{\alpha} \cdot \bar{\xi}, \qquad s = |\bar{y} - \bar{\eta}| \sim \rho - \bar{\beta} \cdot \bar{\eta}. \tag{2.10}$$

Here

$$\bar{\alpha} = \frac{\bar{x}}{\rho}, \qquad \bar{\beta} = \frac{\bar{y}}{\rho} \tag{2.11}$$

are unit vectors along $\bar{x}$, $\bar{y}$, respectively.

Upon using (2.10) and (2.11) in (2.8), we obtain

$$P(\bar{x}, \bar{y}, k) \sim G(\rho, k) G^*(\rho, k) \int_{D^2} Q(\bar{\xi}, \bar{\eta}, k) e^{-ik(\bar{\alpha}\cdot\bar{\xi} - \bar{\beta}\cdot\bar{\eta})} \, d\bar{\xi} \, d\bar{\eta}, \tag{2.12}$$
$$\bar{x}, \bar{y} \in S_\rho, \quad \rho \to \infty.$$

Let $R(\bar{\alpha}, \bar{\beta}, k)$ denote the radiation amplitude, which is defined as the limit of the normalized covariance:

$$R(\bar{\alpha}, \bar{\beta}, k) = \lim_{\substack{\bar{x}, \bar{y} \in S \\ \rho \to \infty}} \frac{P(\bar{x}, \bar{y}, k)}{G(\rho, k) G^*(\rho, k)}. \tag{2.13}$$

By means of (2.13), the asymptotic relation (2.12) yields an integral equation for R:

$$R(\bar{\alpha}, \bar{\beta}, k) = \int_{D^2} Q(\bar{\xi}, \bar{\eta}, k) e^{-ik(\bar{\alpha}\cdot\bar{\xi} - \bar{\beta}\cdot\bar{\eta})} \, d\bar{\xi} \, d\bar{\eta}, \qquad \bar{\alpha}, \bar{\beta} \in U, \tag{2.14}$$

where U is the unit sphere centered at the origin. Let $R = R_1 + iR_2$ and $Q = Q_1 + iQ_2$ so that R_j and Q_j are real, $j = 1, 2$. By taking the real and imaginary parts of (2.14), one obtains the following pair of coupled integral equations:

$$\begin{aligned} R_1(\bar{\alpha}, \bar{\beta}, k) = & \int_{D^2} Q_1(\bar{\xi}, \bar{\eta}, k) \cos k(\bar{\alpha} \cdot \bar{\xi} - \bar{\beta} \cdot \bar{\eta}) \, d\bar{\xi} \, d\bar{\eta} \\ & - \int_{D^2} Q_2(\bar{\xi}, \bar{\eta}, k) \sin k(\bar{\alpha} \cdot \bar{\xi} - \bar{\beta} \cdot \bar{\eta}) \, d\bar{\xi} \, d\bar{\eta}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} R_2(\bar{\alpha}, \bar{\beta}, k) = & \int_{D^2} Q_1(\bar{\xi}, \bar{\eta}, k) \sin k(\bar{\alpha} \cdot \bar{\xi} - \bar{\beta} \cdot \bar{\eta}) \, d\bar{\xi} \, d\bar{\eta} \\ & + \int_{D^2} Q_2(\bar{\xi}, \bar{\eta}, k) \cos k(\bar{\alpha} \cdot \bar{\xi} - \bar{\beta} \cdot \bar{\eta}) \, d\bar{\xi} \, d\bar{\eta}. \end{aligned} \tag{2.16}$$

We note that the integral operator in (2.14) takes, say, an integrable function on D^2 of six dimensions into an integrable (in fact, continuously differentiable) function of the product sphere $U^2 \doteq U \times U$ of a four-dimensional manifold. Therefore the integral equation (2.14), or, equivalently, the equations (2.15) and (2.16), merely specifies a set of admissible solutions, if it exists, instead of determining a unique one. The question of how to provide a logical choice of sensible solutions will be our main concern.

3. Analysis of a discrete model. Before presenting a general theory, it is instructive to work out a simple model in detail. Let us consider the discrete model of N point-sources located at $\bar{x}_j = (a_j, 0, 0)$, $j = 1, 2, \cdots, N$, on the x-axis, where x is the first coordinate of $\bar{x}$. Assuming that the point sources are isotrophic, the source function $\tilde{q}$ takes the form

$$\tilde{q}(\bar{x}, t, \omega) = \sum_{n=1}^{N} \tilde{q}_i(t, \omega) \delta(\bar{x} - \bar{x}_i), \tag{3.1}$$

where $\delta(\bar{x})$ denotes the Dirac delta function and $\tilde{q}_i(t, \omega)$ are independent, centered, second order (weakly) stationary random processes. From (3.1), we form the covari-

ance $\tilde{Q}$ defined by (2.4)

$$\tilde{Q}(\bar{x}, \bar{y}, \tau) = \sum_{j=1}^{N} \tilde{Q}_j(\tau)\delta(\bar{x} - \bar{x}_j)\delta(\bar{y} - \bar{x}_j), \tag{3.2}$$

with

$$\tilde{Q}_j(\tau) = \langle \tilde{q}_j(t)\tilde{q}_j(t+\tau)\rangle, \tag{3.3}$$

or, by taking the Fourier transform (2.8),

$$Q(\bar{x}, \bar{y}, k) = \sum_{j=1}^{N} Q_j(k)\delta(\bar{x} - \bar{x}_j)\delta(\bar{y} - \bar{x}_j). \tag{3.4}$$

A substitution of (3.4) into (2.16) and (2.17) yields

$$R_1(\gamma, k) = \sum_{j=1}^{N} Q_j(k) \cos k\gamma a_j, \qquad R_2(\gamma, k) = \sum_{j=1}^{N} Q_j(k) \sin k\gamma a_j. \tag{3.5}$$

Here $\gamma = \cos\theta_\alpha - \cos\theta_\beta$, and $\theta_\alpha, \theta_\beta$ are the angles between $\bar{\alpha}, \bar{\beta}$ and the x-axis, respectively, and $\bar{\alpha}, \bar{\beta}, \bar{x}_j$ are coplanar.

Suppose that on the measurement surface S, a finite set of radiation data $\{R(\gamma_j, k), j = 1, 2, \cdots, M\}$ is taken. Put

$$\begin{aligned} &r_m(k) = R_1(\gamma_m, k), \qquad r_{M+m}(k) = R_2(\gamma_m, k), \qquad m = 1, 2, \cdots, M. \\ &e_{m,n}(k) = \cos k\gamma_m a_n, \qquad e_{M+m,n}(k) = \sin k\gamma_m a_n, \qquad m = 1, 2, \cdots, M, \\ &\qquad n = 1, 2, \cdots, N. \end{aligned} \tag{3.6}$$

Then (3.5) can be written as

$$\sum_{n=1}^{N} e_{m,n}(k)Q_m(k) = r_m(k), \qquad m = 1, 2, \cdots, 2M. \tag{3.7}$$

Let $E = (e_{m,n})$ be the coefficient matrix and $\tilde{E}$ be the augmented matrix by adjoining the vector (r_m) to the last column of E. It is well-known [8] that (3.7) has a solution if and only if the rank $r(E)$ of E is equal to that of $\tilde{E}$. The solution is unique if $r(E) = N$, which is not true in general. For definiteness, we assume that $r(E) < N$, so that the linear system (3.7) is underdetermined. To single out a "reasonable" solution of (3.7), we may apply one of many versions of the least-square method, mentioned in the previous section. As an alternative, we introduce a selection principle based on the entropy function $\mathcal{T}[Q]$ of noise defined by

$$\mathcal{T}[Q] = -\sum_{n=1}^{N} \int Q_n(k) \ln Q_n(k)\, dk, \tag{3.8}$$

and seek among those solutions of (3.7) the one that maximizes $\mathcal{T}$ (or minimizes $-\mathcal{T}$). According to this procedure, we regard the most favorable solution of (3.7) as being the set of point sources that are distributed in the most chaotic manner. Here, of course, we think of Q_N's as the probability density functions in the information theory [16]. Now the regularized problem can be solved by the method of Lagrangian multipliers [8]. To this end, let $\lambda_m(k)$, $m = 1, 2, \cdots, 2M$, be functions defined for $|k| < \infty$, and let us introduce a residual functional $\mathscr{E}_\lambda[Q]$ of the form

$$\mathscr{E}_\lambda[Q] = \sum_{m=1}^{2M} \int \lambda_m(k)\left[\sum_{n=1}^{N} e_{m,n}(k)Q_n(k) - \gamma_m(k)\right] dk. \tag{3.9}$$

In this case, λ_m are just the Lagrangian multipliers, but, later on, they may be unspecified parameters. Let $\mathscr{L}_\lambda$ denote the Langrangian (functional) defined as

$$\mathscr{L}_\lambda[Q] = \mathscr{T}[Q] + \mathscr{C}_\lambda[Q]. \tag{3.10}$$

For a maximal $\mathscr{T}$, its first variation $\delta\mathscr{L}_\lambda$ with respect to Q must vanish

$$\delta\mathscr{L}_\lambda[Q] = -\sum_{n=1}^{N} \int [1+\ln Q_n]\delta Q_n\, dk + \sum_{m=1}^{2M}\sum_{n=1}^{N} \int \lambda_m e_{m,n}\delta Q_n\, dk = 0, \tag{3.11}$$

which gives the Euler (or Lagrangian) equation

$$\ln Q_n(k) = \sum_{m=1}^{2M} e_{m,n}(k)\lambda_m(k) - 1, \tag{3.12}$$

or

$$Q_n(k) = \exp\left\{\sum_{m=1}^{2M} e_{m,n}(k)\lambda_m(k) - 1\right\}, \qquad n = 1, 2, \cdots, N. \tag{3.13}$$

The above formula determines the source correlations Q_n's provided that λ_m's are known. To find λ_m's, we substitute (3.13) into (3.7) to get

$$\sum_{n=1}^{N} e_{m,n}(k)\exp\left\{\sum_{l=1}^{2M} e_{l,n}(k)\lambda_l(k) - 1\right\} = r_m, \qquad m = 1, 2, \cdots, 2M. \tag{3.14}$$

The above system of nonlinear equations constitutes $2M$ smooth surfaces in λ-space. If these surfaces have a unique point of intersection $\hat{\lambda} = (\hat{\lambda}_1, \cdots, \hat{\lambda}_{2M})$, then the corrsponding source distribution $\hat{Q}_n$ given by (3.13) is the desired solution. As a trivial example, if the auto-correlation $r(k)$, or the mean-square intensity of radiation, at a given point on S is the only measurement taken, then the systems (3.7) and (3.13) reduce to

$$\sum_{n=1}^{N} Q_n(k) = r(k), \tag{3.15}$$

$$Q_n(k) = \exp\{\lambda(k) - 1\}, \qquad n = 1, 2, \cdots, N. \tag{3.16}$$

In this case, we infer that the point-sources are identically distributed with $Q_n(k) = r(k)/N$, $n = 1, 2, \cdots, N$.

In view of (3.14), a regularization by the entropy function $\mathscr{T}[Q]$ leads to a nonlinear system for λ_m's. From the computational viewpoint, a linear system is much preferred. Therefore, for the general problem, the regularization by a quadratic functional $\mathscr{T}$ will be emphasized.

4. A general theory of stochastic inverse source problem. In general we must consider a source distribution whose covariance $\hat{Q}(\bar{x}, \bar{y}, \tau)$ is a generalized function. Without complicating the matter by invoking the theory of generalized functions [17], $\hat{Q}$ will be taken to be a square-integrable function over $D^2 \times \mathbf{R}$. This amounts to restricting our attention to almost continuous source distributions. However, since each generalized function may be regarded as the weak limit of a sequence of ordinary functions, the result in this section will still be valid for singular source distributions, if interpreted in the weak sense.

For convenience, set $\bar{\gamma} = (\bar{\alpha}, \bar{\beta})$, $\bar{z} = (\bar{x}, \bar{y})$, $\bar{\zeta} = (\bar{\xi}, \bar{\eta})$ and

$$K(\bar{\gamma}, \bar{z}, k) = \exp -\{ik(\bar{\alpha}\cdot\bar{\xi} - \bar{\beta}\cdot\bar{\eta})\}, \qquad KQ(\bar{\gamma}, k) = \int_{D^2} K(\bar{\gamma}, \bar{\zeta}, k)Q(\bar{\zeta}, k)\, d\bar{\zeta}. \tag{4.1}$$

Then the integral equation (2.16) can be written as

$$KQ(\bar{\gamma}, k) = R(\bar{\gamma}, k), \qquad \bar{\gamma} \in U^2. \tag{4.2}$$

For each k, the integral operator K takes a complex-valued function $Q \in L^2(D^2)$ into a complex-valued function $R \in L^2(U^2)$.

For the purpose of regularization, let us introduce a structural functional $\mathcal{T}$ of the form

$$\mathcal{T}[Q, Q^*] = \int_{D^2} \int F[Q(\bar{z}, k), Q^*(\bar{z}, k); \nabla Q(\bar{z}, k), \nabla Q^*(\bar{z}, k); \bar{z}, k] \, d\bar{z} \, dk. \tag{4.3}$$

Here the real-valued function F is assumed to be continuous in all of its arguments, continuously differentiable with respect to Q, Q^* and their gradients ∇Q, ∇Q^* in $\bar{z}$. The functional $\mathcal{T}$ plays the role of a constitutive relation in continuum physics in the form of an extremal principle. The functional should be constructed, insofar as feasible, by taking into account all of the a priori information about the noise structure. If none is available, a suitable $\mathcal{T}$ can be chosen for the purpose of smoothing. To define $\mathcal{T}$, let H_1 be the subspace of $H = L^2(D^2 \times \mathbb{R})$ consisting of complex functions Q in H, whose first derivatives $\partial Q/\partial z_j$, $j = 1, 2, \cdots, 6$, are also in H. In view of (4.3) and the ensuing assumptions, the structural functional $\mathcal{T}$ is well-defined in H_1 and its first variation $\delta\mathcal{T}$ is given by [18]

$$\begin{aligned} \delta\mathcal{T}[Q, Q^*] &= \int_{D^2} \int \{(F_Q - \nabla \cdot F_{\nabla Q})\delta Q + (F_{Q^*} - \nabla \cdot F_{\nabla Q^*})\delta Q^*\} \, d\bar{z} \, dk \\ &\quad + \int_{\partial D^2} \int (F_{\nabla Q}\delta Q + F_{\nabla Q^*}Q^*) \cdot \bar{n} \, d\bar{z} \, dk, \end{aligned} \tag{4.4}$$

where $F_Q = \partial F/\partial Q$, $F_{\nabla Q}$ stands for the gradient of F with respect to the complex vector ∇Q, and ∂D^2 is the boundary of D^2 with the outward normal $\bar{n}$. Since F is real, we have $F_Q^* = F_{Q^*}$ and $F_{\nabla Q}^* = F_{\nabla Q^*}$. Now we let $E(u)$ be a smooth function of a complex variable u such that $|E(u)|$ is convex and $E(0) = 0$, e.g. $E(u) = u, uu^*, \cdots$, etc. For every complex $\lambda(\bar{\gamma}, k) \in L^2(U^2 \times \mathbb{R})$, the residual functional $\mathcal{E}_\lambda$ is defined as

$$\begin{aligned} \mathcal{E}_\lambda[Q] = \int_{U^2} \int &\{\lambda(\bar{\gamma}, k)E[KQ(\bar{\gamma}, k) - R(\bar{\gamma}, k)] \\ &+ \lambda^*(\bar{\gamma}, k)E^*[KQ(\bar{\gamma}, k) - R(\bar{\gamma}, k)]\} \, d\bar{\gamma} \, dk, \end{aligned} \tag{4.5}$$

which, by construction, is real-valued.

As in the discrete model, we introduce the Lagrangian functional

$$\mathcal{L}_\lambda[Q] = \mathcal{T}[Q, Q^*] + \mathcal{E}_\lambda[Q], \tag{4.6}$$

but the parametric function λ need not be interpreted as a Lagrangian multiplier. In the regularization process, we propose the principle of extremal Lagrangian with or without side conditions. To proceed, the existence of extrema will always be assumed. For a stationary $\mathcal{L}_\lambda$, we get, by invoking (4.4)–(4.6)

$$\delta\mathcal{L}_\lambda[Q] = \delta\mathcal{T}[Q, Q^*] + \delta\mathcal{E}_\lambda[Q] = 0, \tag{4.7}$$

or

$$(4.8)\quad \begin{aligned}&\int_{D^2}\int \{(F_Q-\nabla\cdot F_{\nabla Q})\delta Q+(F_{Q^*}-\nabla\cdot F_{\nabla Q^*})\delta Q^*\}\,d\bar{z}\,dk\\ &+\int_{D^2}\int_{U^2}\int \{\lambda K\dot{E}(KQ-R)\delta Q+\lambda^* K^*\dot{E}^*(KQ-R)\delta Q^*\}\,d\bar{z}\,d\bar{\gamma}\,dk\\ &+\int_{\partial D^2}\int (F_{\nabla Q}\cdot\bar{n}\delta Q-F_{\nabla Q^*}\cdot\bar{n}\delta Q^*)\,d\bar{z}\,dk=0,\end{aligned}$$

in which $\dot{E}(u)=dE(u)/du_1$ and $u_1=\mathrm{Re}\{u\}$. Since δQ and δQ^* are two arbitrary elements of H_1, the equation (4.8) implies the Euler differential equation in D^2 [8]

$$(4.9)\quad \begin{aligned}&\nabla\cdot F_{\nabla Q}(Q,\nabla Q,\bar{z},k)-F_Q(Q,\nabla Q,\bar{z},k)\\ &=\int_{U^2}\lambda(\bar{\gamma},k)K(\bar{\gamma},\bar{z},k)\dot{E}[KQ(\bar{\gamma},k)-R(\bar{\gamma},k)]\,d\bar{\gamma},\end{aligned}$$

and the natural boundary condition

$$(4.10)\quad F_{\nabla Q}(Q,\nabla Q,\bar{z},k)|_{\partial D^2}=0.$$

Here, in the arguments of F_Q and $F_{\nabla Q}$, the conjugate variables Q^* and ∇Q^* are omitted for simplicity, and the equations corresponding to δQ^* are simply the complex conjugate of (4.9) and (4.10) and, hence, will not be needed.

If no side conditions are imposed, the parametric function λ should be suitably chosen so that the boundary-value problem (4.9) and (4.10) has a unique, stable solution Q_λ in H_1. Otherwise, e.g. regarding (4.2) as a side condition, we shall consider λ to be a Langrangian multiplier. In either case, Q_λ will be accepted as a generalized solution to the ill-posed problem. To be specific and to demonstrate the flexibility of the proposed scheme, we shall present three examples which give different interpretations of the general principle.

5. Examples.

A. *Continuous source with the least noise intensity.* Perhaps the simplest choice of the structural functional $\mathcal{T}$ is the total intensity of Q defined as

$$(5.1)\quad \mathcal{T}[Q,Q^*]=\frac{1}{2}\int_{D^2}\int Q(\bar{z},k)Q^*(\bar{z},k)\,d\bar{z}\,dk.$$

Among all solutions of (4.2), we wish to single out the one that minimizes $\mathcal{T}$. The problem of minimizing $\mathcal{T}$ subject to the constraint (4.2) can be solved by the method of the Lagrangian multiplier λ with $E(u)=u$. In this case the Euler's equation (4.9) becomes simply

$$(5.2)\quad Q_\lambda(\bar{z},k)=-\int_{U^2}\lambda(\bar{\gamma},k)K(\bar{\gamma},\bar{z},k)\,d\bar{\gamma}=-K^*\lambda(\bar{z},k),$$

where K^* designates the adjoint of K.

Let us introduce the integral operator $\hat{K}$ defined as

$$(5.3)\quad \hat{K}\lambda(\bar{\gamma},k)=KK^*\lambda(\bar{\gamma},k)=\int_{D^2}\int_{U^2}K(\bar{\gamma},\bar{z},k)\lambda(\bar{\gamma}',k)K^*(\bar{\gamma}',\bar{z},k)\,d\bar{z}\,d\bar{\gamma}'.$$

Upon substituting (5.2) into (4.2), we obtain the integral equation for λ, in the

operator notation

$$\hat{K}\lambda = -R. \tag{5.4}$$

Note the $\hat{K}$ defined above is a self-adjoint, Hilbert–Schmidt operator on $L^2(U^2)$. If the radiation amplitude R is in the range of $\hat{K}$, the Fredholm integral equation (5.4) has a unique solution given by

$$\hat{\lambda} = -\hat{K}^{-1}R = -(KK^*)^{-1}R. \tag{5.5}$$

Alternatively R belongs to the domain of the inverse $\hat{K}^{-1}$ provided that [19]

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2}\left|\int_{D^2} R(\bar{z}, k)\,d\bar{z}\right|^2 < \infty, \tag{5.6}$$

where $\{\mu_n\}$ and $\{\phi_n\}$ are the eigenvalues and eigenfunctions of K respectively.

In view of (5.5), the equation (5.2) yields a generalized (or pseudo) solution

$$Q_{\hat{\lambda}} = K^*(KK^*)^{-1}R = TR. \tag{5.7}$$

It is interesting to compare the above result $T = K^*(KK^*)^{-1}$ with the pseudo inverse $T_g = (K^*K)^{-1}K^*$ of K [9], which can be computed easily by applying K^* to the equation (4.2) and inverting the resulting equation. This pseudo inverse has the statistical interpretation as the best linear estimator which minimizes the mean-square data noises. It is not surprising that two different selection principles lead to two distinct generalized solutions for a given image function R.

B. *Continuous source with a potential.* Next we consider the source correlation Q as the equilibrium state of a fictitious physical variable in the continuous body D^2 such that the structural functional corresponds to the total "potential energy"

$$\mathcal{T}[Q, Q^*] = \frac{1}{2}\int_{D^2}\int \{-a(\bar{z}, k)\nabla Q \cdot \nabla Q^* + b(\bar{z}, k)QQ^*\}\,d\bar{z}\,dk, \tag{5.8}$$

where, for each k, a and b are real, positive, continuous functions, and a is continuously differentiable in D^2. At equilibrium, the potential energy is minimum under the condition (4.2). Similar to the previous example, one is led to the regularized boundary value problem for the Euler equation

$$\nabla \cdot (a\nabla Q) + bQ = -\int_{U^2} \lambda(\bar{\gamma}, k)K(\bar{\gamma}, \bar{z}, k)\,d\bar{\gamma}, \tag{5.9}$$

$$\nabla Q \cdot \bar{n}\Big|_{\partial D^2} = \frac{\partial Q}{\partial n}\Big|_{\partial D^2} = 0. \tag{5.10}$$

Let $N(\bar{z}, \bar{\zeta}, k)$ denote the Neumann function [15] for the above boundary-value problem. Then this problem is equivalent to

$$Q(\bar{z}, k) = -NK^*\lambda(\bar{z}, k) = -\int_{D^2}\int_{U^2} N(\bar{z}, \bar{\xi}, k)K(\bar{\gamma}, \bar{z}, k)\lambda(\bar{\gamma}, k)\,d\bar{z}\,d\bar{\gamma}, \tag{5.11}$$

where λ is as yet to be determined. To this end, we use (5.11) in (4.2) to get the operator equation

$$\hat{K}\lambda(\bar{\gamma}, k) = (KNK^*)\lambda(\bar{\gamma}, k) = -R(\bar{\gamma}, k). \tag{5.12}$$

Again $\hat{K}$ defined above is a self-adjoint, Hilbert–Schmidt operator from $L^2(U^2)$ into itself. If R satisfies the solvability condition (5.6), by inverting (5.12), the equation

(5.11) gives the generalized solution

$$Q_\lambda(\bar z, k) = NK^*(KNK^*)^{-1}R(\bar z, k). \tag{5.13}$$

Of course, when $a \equiv 0$, $b \equiv \frac{1}{2}$, we have $N = I$, an identity operator, and the result (5.7) follows as a special case.

We remark that the equations (5.3) and (5.12) for λ are unstable. In actual computation, an additional smoothing procedure, such as Tihonov's is needed to ensure stability. The next example will illustrate this procedure.

C. *Axisymmetric line source.* As shown in [20], for an axisymmetric line source distributed continuously along the x-axis in the interval $I = [0, 1]$, the integral equation (4.2) reduces to

$$\begin{aligned} KQ(\bar\gamma, k) &= \int_{I^2} \exp[-ik(\alpha x - \beta y)]Q(x, y, k)\,dx\,dy \\ &= R(\bar\gamma, k), \end{aligned} \tag{5.14}$$

where $\bar z = (x, y)$, $\bar\gamma = (\alpha, \beta)$, and α, β are the direction cosines of the lines of sight with respect to the x-axis as indicated in § 2. Assuming the condition (5.6) for the data R holds, the integral equation (5.14) has a unique solution, but the solution does not depend continuously on data. For computational purposes, the general procedure outlined before can be employed as a smoothing scheme. In particular, if we choose the residual function $E(u) = \frac{1}{2}uu^*$ and $\lambda \equiv 1/\varepsilon$, where $\varepsilon > 0$ is a small parameter, the Euler's equation (4.9) reads

$$\varepsilon\{\nabla \cdot F_{\nabla Q} - F_Q\} = (K^*K)Q - K^*R, \tag{5.15}$$

and

$$\left.\frac{\partial Q}{\partial n}\right|_{\partial I^2} = 0. \tag{5.16}$$

For instance, let us set $F[Q, Q^*] = \frac{1}{2}\{-a\nabla Q \cdot \nabla Q^* + bQQ^*\}$ as in the previous example. The equation (5.15) takes the form

$$\varepsilon\{\nabla \cdot (a\nabla Q) + bQ\} = K^*R - (K^*K)Q. \tag{5.17}$$

Thereby we have arrived at Tihonov's regularization as a special case, where $\mathcal{T}$ is termed as the smoothing functional. The boundary-value problem (5.16) and (5.17) is well-posed because it is equivalent to the Fredholm integral equation of second kind:

$$(NK^*K)Q - \varepsilon Q = (NK^*)R. \tag{5.18}$$

In practice the smoothed problem may be solved by finite deferences or finite elements methods, where the value of ε is usually chosen by trial and error. Numerical examples can be found in the paper [20]. However a systematic way of choosing the parameter ε was discussed in detail by Franklin [21] in one of his interesting papers on ill-posed problems.

Appendix.

A.1. Regularization of a general inverse radiation problem. The regularization procedure presented in § 4 does not limit its application to the wave equation. In fact the method applies to other linear evolution equations provided that the far-field radiation data is stationary and can be properly defined. Here we shall treat a linear system of hyperbolic equations of first order. Let $\bar u = (u_1, u_2, \cdots, u_n)$ be a vector field

satisfying the following system of equations

$$M\tilde{u} = \frac{\partial \tilde{u}}{\partial t} + A(\nabla_x)\tilde{u} = \tilde{q}(\bar{x}, t, \omega), \qquad \bar{x} \in \mathbb{R}^3, \tag{A.1}$$

where $A(\bar{x})$ is a matrix-valued, linear function of $\bar{x}$, and $\tilde{q}$ is a vector-valued, stationary random function, describing the source distribution in a region D. Suppose that $\tilde{G}$ denote the Riemann radiation matrix [15] satisfying the adjoint equation

$$M^*\tilde{G}(\bar{x}-\bar{\xi}, t-s) = \delta(t-s)\delta(\bar{x}-\bar{\xi})I, \tag{A.2}$$

where I signifies the unit matrix. The radiation field of (A.2) is then given by

$$\tilde{u}(\bar{x}, t, \omega) = \tilde{G}\tilde{q}(\bar{x}, t, \omega) = \int\int_D \tilde{G}(\bar{x}-\bar{\xi}, t-s)\tilde{q}(\bar{\xi}, s, \omega)\, ds\, d\bar{\xi}. \tag{A.3}$$

If $\langle \tilde{q} \rangle = 0$, we form the covariance matrices (or tensors) of $\tilde{u}$ and $\tilde{q}$ defined as

$$\tilde{P}(\bar{x}, \bar{y}, \tau) = \langle \tilde{u}(\bar{x}, t+\tau)\tilde{u}^*(\bar{y}, t)\rangle, \qquad \tilde{Q}(\bar{x}, \bar{y}, \tau) = \langle \tilde{q}(\bar{x}, t+\tau)\tilde{q}^*(\bar{y}, t)\rangle. \tag{A.4}$$

In view of (A.3) and (A.4), we obtain

$$\tilde{P}(\bar{x}, \bar{y}, \tau) = \int\int\int_{D^2} \tilde{G}(\bar{x}-\bar{\xi}, t+\tau-s)Q(\bar{\xi}, \bar{\eta}, s-s')\tilde{G}^*(\bar{y}-\bar{\eta}, t-s)\, ds\, ds'\, d\bar{\xi}\, d\bar{\eta}. \tag{A.5}$$

The above integral can be shown to be independent of t by the Fourier transform (2.7) in τ (setting $c = 1$) and by a change of variable

$$P(\bar{x}, \bar{y}, k) = \int_{D^2} [G \otimes G](\bar{x}-\bar{\xi}, \bar{y}-\bar{\eta}, k)Q(\bar{\xi}, \bar{\eta}, \tau)\, d\bar{\xi}\, d\bar{\eta} \tag{A.6}$$

where $(G \otimes G)Q = GQG^*$ denotes a tensor product. As in § 2, given the radiation data on the spherical surface S_ρ, we define the normalized radiation covariance similar to (2.14)

$$R(\bar{\alpha}, \bar{\beta}, k) = \lim_{\substack{\bar{x}, \bar{y} \in S_\rho \\ \rho \to \infty}} [G \otimes G]^{-1}(\bar{x}, \bar{y}, k)P(\bar{x}, \bar{y}, k). \tag{A.7}$$

Corresponding to the kernel K given by (4.1), we have

$$K(\bar{\gamma}, \bar{\zeta}, k) = \lim_{\substack{\bar{x}, \bar{y} \in S_\rho \\ \rho \to \infty}} [G \otimes G]^{-1}(\bar{x}, \bar{y}, k)[G \otimes G](\bar{x}-\bar{\xi}, \bar{y}-\bar{\eta}, k). \tag{A.8}$$

Again we arrive at the familiar equation

$$\int_{D^2} K(\bar{\gamma}, \bar{\zeta}, k)Q(\bar{\zeta}, k)\, d\bar{\zeta} = R(\bar{\gamma}, k). \tag{A.9}$$

Thus it becomes clear that our regularization procedure introduced in § 4 works in the general case.

A.2. A stochastic well-posed extension. Without being specific, consider the stochastic linear equation

$$p = Aq, \tag{A.10}$$

where A is bounded linear operator taking a random vector $q \in H_1$ into another random vector $p \in H$, where H, H_1 are two complex Hilbert spaces. In particular the

equation (A.10) may represent the far-field relation of (2.2) in the spectral (k) domain. Suppose that the random data p is distorted by a noise d in H. Instead of (A.10), we must consider the perturbed equation

(A.11) $$Aq = p + d.$$

Assuming that all random vectors are centered, the covariances P, Q, D, R_{pq}, R_{pd}, R_{qd} are defined in terms of the inner products $(\cdot, \cdot)$, $[\cdot, \cdot]$ in H and H_1, respectively. For example

(A.12) $$(Pf, g) = \langle (p, f)(p, g) \rangle, \qquad [Qf_1, g_1] = \langle [q, f_1][q, g_1] \rangle,$$
$$(R_{pq} g_1, f) = \langle (p, f)[q, g_1] \rangle, \text{ etc.},$$

where $f, g \in H$ and $f_1, g_1 \in H_1$.

We wish to estimate q by a bounded linear transformation B of p, i.e.

(A.13) $$\hat{q} = Bp,$$

so that the mean-square error is a minimum

(A.14) $$\varepsilon^2 = \min_{f_1 \in H_1} \langle [(q - \hat{q}) f_1, f_1] \rangle.$$

Following Franklin [5], we use (A.13) in (A.14) to determine B when the minimum value ε^2 is attained. By "completing the square", it can be shown that

(A.15) $$B = R_{pq} P^{-1}.$$

Now we make use of (A.11) to get

(A.16) $$(AQf_1, g) = (R_{pq} f_1, g) + (R_{dq} f_1, g),$$

or

(A.17) $$AQ = R_{pq} + R_{dq}.$$

An elimination of R_{pd} from (A.15) and (A.17) gives

(A.18) $$B = (AQ - R_{dq}) P^{-1}.$$

On the other hand (A.13) implies that

(A.19) $$\hat{Q} = BPB^*.$$

Upon using (A.18) in (A.19) with Q replaced by $\hat{Q}$, we obtain a nonlinear well-posed equation for $\hat{Q}$:

(A.20) $$\hat{Q} = (A\hat{Q} - R_{dq}) P^{-1} (A\hat{Q} - R_{dq})^*,$$

where R_{dq} must be suitably chosen a priori. In contrast with Franklin's approach, R_{pq} is eliminated from (A.15) to yield a linear equation, using (A.11) in terms of P and R_{pd}, which are to be chosen judiciously.

REFERENCES

[1] M. J. LIGHTHILL, *On source generated aerodynamically II, Turbulence as a source of sound*, Proc. Roy. Soc. A., 222 (1954), pp. 1–32.

[2] H. S. RIBNER, *Theory of Two-Point Correlations of Jet Noise*, NASA TN D-8330, 1976.

[3] J. HADAMARD, *Lectures on Cauchy's Problems in Linear Partial Differential Equations*, Dover, New York, 1952.

[4] A. N. TIHONOV, *Solution of incorrectly formulated problems and the regularization method*, Sov. Math. Dokl., 4 (1963), pp. 1035–1038.

[5] J. N. FRANKLIN, *Well-posed stochastic extensions of ill-posed linear problems*, J. Math. Anal. Appl., 31 (1970), pp. 682–716.

[6] G. BACKUS, *Inference from incomplete and inaccurate data I*, Proc. Nat. Acad. Sci., 65 (1970), pp. 1–8.

[7] K. MILLER, *Least squares methods for ill-posed problems with a prescribed bound*, this Journal, 1 (1970), pp. 52–74.

[8] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953.

[9] C. R. RAO AND S. K. MITRA, *Generalized Inverse of Matrices and Its Applications*, John Wiley, New York, 1971.

[10] A. CARASSO AND A. P. STONE, eds., *Improperly Posed Boundary Value Problems*, Pitman, London, 1975.

[11] M. M. LAVRENTIEV, *Some Improperly Posed Problems of Mathematical Physics*, Springer Tracts in Natural Philosophy, vol. II, Springer-Verlag, New York, 1967.

[12] R. G. NEWTON, *Scattering Theory of Waves and Particles*, McGraw-Hill, New York, 1966.

[13] J. B. KELLER, *Inverse problems*, Amer. Math. Monthly, 83 (1976), pp. 107–118.

[14] W. BLEISTEIN AND J. K. COHEN, *Nonuniqueness in the inverse source problem in acoustics and electromagnetics*, J. Math. Phys., 18 (1977), pp. 184–201.

[15] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. II, Interscience, New York, 1962.

[16] A. I. KHINCHIN, *Mathematical Foundation of Information Theory*, Dover, New York, 1957.

[17] I. M. GEL'FAND AND G. E. SHILOV, *Generalized Functions*, vol. I, Academic Press, New York, 1964.

[18] S. G. MIKHLIN, *The Problem of the Minimum of a Quadratic Functional*, Holden-Day, San Francisco, 1965.

[19] S. G. MIKHLIN, *Integral Equations*, Pergamon Press, New York, 1957.

[20] L. MAESTRELLO AND C. H. LIU, *Jet noise source distribution from far-field correlations*, AIAA J., 15 (1977), pp. 71–72.

[21] J. N. FRANKLIN, *On Tikhonov's method for ill-posed problems*, Math. Comput., 28 (1974), pp. 889–903.